

# Quantum and classical stochastic dynamics: Exactly solvable models by supersymmetric methods\*

Georg Junker

Institut für Theoretische Physik, Universität Erlangen-Nürnberg,  
Staudtstr. 7, D-91058 Erlangen, Germany

*E-mail:* junker@theorie1.physik.uni-erlangen.de

## Abstract

A supersymmetric method for the construction of so-called conditionally exactly solvable quantum systems is reviewed and extended to classical stochastic dynamical systems characterized by a Fokker-Planck equation with drift. A class of drift-potentials on the real line as well as on the half line is constructed for which the associated Fokker-Planck equation can be solved exactly. Explicit drift potentials, which describe mono-, bi-, meta- or unstable systems, are constructed and their decay rates and modes are given in closed form.

## 1 Introduction

Many physical phenomena of nature are characterized by some basic differential equations. For example, quantum-mechanical phenomena are described by Schrödinger's equation, which dictates the dynamics of some quantum system represented by a Hamilton operator. One is therefore primarily interested in finding all eigenvalues and eigenstates of such Hamiltonians. As a consequence, finding a large class of, in this sense, analytically exactly solvable quantum systems is an important goal and this search has already been initiated by Schrödinger using the so-called factorization method [1, 2]. This factorization method is basically equivalent to Darboux's method [3] already invented 1882 and applied to Sturm-Liouville problems. Recently, these methods have attracted new attention [4], in particular in connection

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with supersymmetric (SUSY) quantum mechanics [5]. For a recent approach and further references see, for example, [6, 7, 8].

Somehow unnoticed by those working on the quantum mechanical problem the same basic ideas have been utilized in constructing exactly solvable classical stochastic systems described by a Fokker-Planck equation [9, 10, 11]. In fact, the Fokker-Planck equation can be put into the form of an imaginary-time Schrödinger equation exhibiting SUSY. Therefore, exactly solvable SUSY quantum Hamiltonians also provide exactly solvable Fokker-Planck-type systems. Whereas until now these methods have been applied to mono- and bistable systems [9, 10, 11], they correspond to a situation with unbroken SUSY, we present here also examples with broken SUSY giving rise to meta- or unstable drift potentials.

In the next two sections we briefly review the basic concepts of SUSY quantum mechanics and its close connection with the Fokker-Planck equation [5, 12]. Section 4 reviews our recent approach [6, 7, 8] towards a construction of the most general class of SUSY partners of a given quantum mechanical Hamiltonian. This approach is applied to the Fokker-Planck equation and allows to construct new drift potentials with known decay rates and modes. Two examples are discussed in some detail. The first one is related to the linear harmonic oscillator having unbroken SUSY and leads to bistable double-well or monostable single-well drift potentials already found by Hongler and Zheng [9]. The second example is the class related to the radial harmonic oscillator with broken SUSY. Here the drift potential is either metastable or unstable.

## 2 Supersymmetric quantum mechanics

To begin with, let us briefly review the basics of SUSY quantum mechanics or, to be more precise, Witten's model of SUSY quantum mechanics [5]. This model consists of a pair of standard Schrödinger Hamiltonians (units are such that  $\hbar = m = 1$ )

$$H_{\pm} = -\frac{1}{2} \frac{d^2}{dx^2} + V_{\pm}(x), \quad V_{\pm}(x) = \frac{1}{2} \left( W^2(x) \pm W'(x) \right). \quad (1)$$

Evidently, this pair is completely characterized by the so-called SUSY potential  $W$ , which is assumed to be real-valued and an at least once differentiable function inside the configuration space. This configuration space will be either the real line  $\mathbb{R}$  or the positive half line  $\mathbb{R}^+$ . Thus the above Hamiltonians act on a suitable linear space of square-integrable functions over  $\mathbb{R}$  and  $\mathbb{R}^+$ , respectively. With the help of the supercharge operators

$$A = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + W(x) \right), \quad A^{\dagger} = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + W(x) \right) \quad (2)$$

the above pair of SUSY-partner Hamiltonians factorizes,  $H_+ = AA^{\dagger} \geq 0$ ,  $H_- = A^{\dagger}A \geq 0$ , and obviously obeys the intertwining relations  $AH_- = H_+A$  and  $H_-A^{\dagger} = A^{\dagger}H_+$ . As a consequence  $H_+$  and  $H_-$  are essentially isospectral. In other words,

their strictly positive energy eigenvalues coincide and the corresponding eigenstates are related via the supercharges. However, there may exist an additional vanishing eigenvalue for one of these Hamiltonians. In this case SUSY is said to be unbroken and by standard convention [5] this additional eigenvalue is assumed to belong to  $H_-$ . If such a state does not exist SUSY is said to be broken. To summarize, for unbroken SUSY we have

$$E_0^- = 0, \quad E_{n+1}^- = E_n^+ > 0, \quad \psi_0^-(x) = \psi_0^-(0) \exp\left\{-\int_0^x dz W(z)\right\}, \quad (3)$$

$$\psi_{n+1}^- = (E_n^+)^{-1/2} A^\dagger \psi_n^+, \quad \psi_n^+ = (E_{n+1}^-)^{-1/2} A \psi_{n+1}^-,$$

whereas for broken SUSY these relations read

$$E_n^- = E_n^+ > 0, \quad \psi_n^- = (E_n^+)^{-1/2} A^\dagger \psi_n^+, \quad \psi_n^+ = (E_n^-)^{-1/2} A \psi_n^-, \quad (4)$$

Here we have denoted the eigenfunctions and eigenvalues of  $H_\pm$  by  $\psi_n^\pm$  and  $E_n^\pm$ , respectively. That is,

$$H_\pm \psi_n^\pm = E_n^\pm \psi_n^\pm, \quad n = 0, 1, 2, \dots, \quad (5)$$

and we also note that throughout this paper we will consider, without loss of generality, quantum systems with a purely discrete spectrum.

We conclude this section by noting that for broken as well as unbroken SUSY one can obtain the complete spectral information of one Hamiltonian, say  $H_-$ , if the eigenvalues and eigenstates of the corresponding partner, here  $H_+$ , and the SUSY potential  $W$  are known.

### 3 Classical stochastic dynamics

As mentioned in the Introduction we are also interested in classical systems with a stochastic dynamics governed by the Fokker-Planck equation

$$\frac{\partial}{\partial t} m_t(x, x_0) = \frac{1}{2} \frac{\partial^2}{\partial x^2} m_t(x, x_0) + \frac{\partial}{\partial x} \left( U'(x) m_t(x, x_0) \right). \quad (6)$$

Here  $m_t(x, x_0)$  denotes the transition-probability density of a macroscopic degree of freedom to be found at time  $t$  at position  $x$  if it initially has been at  $x_0$ , i.e.  $m_0(x, x_0) = \delta(x - x_0)$ . This degree of freedom is subjected to an external force characterized by the real-valued drift potential  $U$  and to a stochastic random force (white noise) resulting in the diffusive term on the right-hand side of (6) with diffusion constant set equal to  $1/2$ . Making the ansatz [12]

$$m_t(x, x_0) = e^{-U(x)} K_t(x, x_0) \quad (7)$$

leads to

$$-\frac{\partial}{\partial t} K_t(x, x_0) = \left[ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} U'^2(x) - \frac{1}{2} U''(x) \right] K_t(x, x_0), \quad (8)$$

which may be interpreted as an imaginary-time Schrödinger equation for the SUSY Hamiltonian  $H_-$  with a SUSY potential given by the first derivative of the drift potential,  $W = U'$ . Hence, the desired transition-probability density is given via the Euclidean propagator for  $H_-$ :

$$m_t(x, x_0) = \exp\{U(x_0) - U(x)\} \langle x | \exp\{-tH_-\} | x_0 \rangle. \quad (9)$$

It is obvious that the decay modes and decay rates of this classical stochastic dynamical system are related to the eigenfunctions and eigenvalues of  $H_-$ . To be more explicit, let us consider the cases of unbroken and broken SUSY separately.

For unbroken SUSY the ground-state energy of  $H_-$  vanishes and as a consequence there exists a stationary, i.e. time-independent, probability distribution. Noting that  $\psi_0^-(x) = \psi_0^-(x_0) \exp\{U(x_0) - U(x)\}$  the transition-probability density can be put into the form

$$m_t(x, x_0) = [\psi_0^-(x)]^2 + \exp\{U(x_0) - U(x)\} \sum_{n=1}^{\infty} e^{-tE_n^-} \psi_n^-(x) \psi_n^-(x_0), \quad (10)$$

which clearly shows that the strictly positive eigenvalues of  $H_-$  and the associated eigenfunctions characterize the decay rates and decay modes of the system. In addition the ground-state wavefunction represents the stationary distribution. That is, a stable system is characterized by an unbroken SUSY.

In the case of broken SUSY, by definition,  $H_-$  does have strictly positive eigenvalues only and cannot lead to a stationary distribution. In other words, broken SUSY is related to unstable systems. Here the transition probability density has the form

$$m_t(x, x_0) = \exp\{U(x_0) - U(x)\} \sum_{n=0}^{\infty} e^{-tE_n^-} \psi_n^-(x) \psi_n^-(x_0). \quad (11)$$

Finally, we mention that in both cases we may invert the drift potential,  $U \rightarrow -U$ , which amounts in replacing  $H_-$  by  $H_+$ . Hence, due to the presence of SUSY the inverted drift potential has the same decay rates as the original one. Only in the case of unbroken SUSY a stable system becomes unstable upon inversion. This is expressed in the fact that the vanishing eigenvalue  $E_0^- = 0$  is missing in the spectrum of  $H_+$ . The decay modes of the original and the inverted drift potential are also clearly related by the SUSY transformations (3) and (4), respectively.

### 4 Designing exactly solvable models

From the discussion in the previous two sections it is clear that once the spectral data, that is, the eigenvalues and eigenfunctions, of  $H_+$  are known we also know these data for the corresponding superpartner  $H_-$  and in turn the decay rates and modes for the Fokker-Planck equation with drift potential  $U(x) = \int_{x_0}^x dz W(z)$ . In order to construct new exactly solvable systems we will search for the most general class of superpartners associated with a given exactly solvable quantum Hamiltonian

$H_+$ . Such a construction method has recently been given [6, 7, 8] and we briefly review it here.

In order to find the most general class of SUSY partners of a given SUSY Hamiltonian we make the following ansatz

$$W(x) = \Phi(x) + \frac{u'(x)}{u(x)}, \quad (12)$$

where  $\Phi$  is a known SUSY potential belonging to some shape-invariant (i.e. exactly solvable) potential. See, for example, Table 5.1. in ref. [5]. If we now assume that  $u$  is a solution of

$$u''(x) + 2\Phi(x)u'(x) - bu(x) = 0, \quad b \in \mathbb{R}, \quad (13)$$

we find that

$$V_+(x) = \frac{1}{2}\Phi^2(x) + \frac{1}{2}\Phi'(x) + \frac{b}{2}. \quad (14)$$

By construction  $V_+$  is up to the additive constant  $b/2$  shape-invariant and, therefore, the eigenvalues and eigenfunctions of  $H_+$  are exactly known. The corresponding partner potential can be put into the form

$$V_-(x) = \frac{1}{2}\Phi^2(x) - \frac{1}{2}\Phi'(x) + \frac{u'(x)}{u(x)} \left( 2\Phi(x) + \frac{u'(x)}{u(x)} \right) - \frac{b}{2}, \quad (15)$$

which, in general, is not shape-invariant and thus a *new* exactly solvable quantum mechanical potential. Similarly we may construct also a *new* drift potential

$$U(x) = \int_{x_0}^x dz \Phi(z) + \log u(x) \quad (16)$$

for which the associated decay rates and modes are known exactly.

Let us note here that we cannot take an arbitrary solution of (13) for the construction of new quantum-mechanical and drift potentials. In order to circumvent domain questions of the operators involved we will take only those solutions of (13) into account which are strictly positive. Hence, no singularities inside the configuration space will appear in  $V_-$  and  $U$ . For this reason the new Schrödinger potentials  $V_-$  obtained in this way have been called conditionally exactly solvable [6]. In the following two subsections we will present two examples with unbroken and broken SUSY, respectively. For further details and examples see [8].

#### 4.1 A system with unbroken SUSY

The first example we are going to present is characterized by a linear SUSY potential,  $\Phi(x) = x$ , with configuration space given by the real line,  $x \in \mathbb{R}$ . This SUSY potential gives rise to the harmonic-oscillator potential

$$V_+(x) = \frac{1}{2}(x^2 + b + 1) \quad (17)$$

and the corresponding eigenvalues and eigenfunctions of  $H_+$  are

$$E_n^+ = n + b/2 + 1, \quad \psi_n^+(x) = [\sqrt{\pi} 2^n n!]^{-1/2} H_n(x) \exp\{-x^2/2\}. \quad (18)$$

Here  $H_n$  denotes a Hermite polynomial of order  $n$ . In this case the general solution of (13) can be expressed in terms of confluent hypergeometric functions,

$$u(x) = {}_1F_1\left(-\frac{b}{4}, \frac{1}{2}, -x^2\right) + \beta x {}_1F_1\left(\frac{2-b}{4}, \frac{3}{2}, -x^2\right) \quad (19)$$

and is strictly positive for  $b > -2$  and  $|\beta| < 2\Gamma(\frac{b}{4} + 1)/\Gamma(\frac{b+2}{4})$ , cf. ref. [7, 8]. The corresponding partner potential is given by

$$V_-(x) = \frac{1}{2}x^2 - \frac{b+1}{2} + \frac{u'(x)}{u(x)} \left[ 2x + \frac{u'(x)}{u(x)} \right] \quad (20)$$

and plots of for various values of the parameters  $b$  and  $\beta$  can be found in Figure 1 of [7] and Figures 1 and 2 of [8]. Here we only note that for large  $|x|$  this potential becomes asymptotically that of a harmonic oscillator. For values of  $b$  close to the lower bound  $-2$  this potential exhibits two double-wells near the origin. Whereas for larger values of  $b$  these double wells merge to a single well at the origin. For  $\beta = 0$  the potential  $V_-$  is symmetric about  $x = 0$ . This symmetry is broken for non-vanishing  $\beta$ . Noting that SUSY is unbroken for all allowed values of the parameters, the spectral properties of the corresponding Hamiltonian  $H_-$  are easily obtained from (3):

$$E_0^- = 0, \quad E_{n+1}^- = E_n^+ = n + b/2 + 1, \quad \psi_0^-(x) = \frac{\psi_0^-(0)}{u(x)} \exp\{-x^2/2\},$$

$$\psi_{n+1}^-(x) = \frac{\exp\{-x^2/2\}}{[\sqrt{\pi} 2^{n+1} n!(n + b/2 + 1)]^{1/2}} \left( H_{n+1}(x) + H_n(x) \frac{u'(x)}{u(x)} \right). \quad (21)$$

These results also allow a complete study of the Fokker-Planck equation for the drift potential

$$U(x) = \frac{1}{2}x^2 + \log u(x). \quad (22)$$

In fact, this drift potential is stable and has a stationary distribution given by  $[\psi_0^-(x)]^2$ . The decay rates and decay modes are given explicitly by  $E_{n+1}^-$  and  $\psi_{n+1}^-$  in (21). Plots of this drift potential for various values of the parameters can be found in Figures 1 of [9]. Here we again briefly mention that for  $b$  close to its lower limit  $U$  has the shape of a bistable (double-well) potential being symmetric about the origin for  $\beta = 0$ . For larger values of  $b$  this drift potential develops also a stable single-well shape.

To conclude this subsection let us mention that the results presented here are not new. They have first been derived by Hongler and Zheng [9] in 1982, which was even before the work of Mielnik [4] who was searching for more general factorizations of the harmonic oscillator Hamiltonian. His results can be viewed as special

cases of those presented here. The advantage of the present approach is that it can be applied not only to the harmonic oscillator-like systems but to all exactly solvable ones [8]. In particular, there exist also shape-invariant quantum systems which allow for a broken SUSY [5]. That is, one may be able to design drift potentials which are metastable and can be solved exactly. To our knowledge, such exactly solvable systems are not available so far. Except, of course, the rather trivial case of a piecewise linear drift potential. In the next subsection we are going to present an exactly solvable metastable drift potential, which is related to the radial harmonic oscillator Hamiltonian.

#### 4.2 A system with broken SUSY

In this subsection we will consider the case where the SUSY potential is given by

$$\Phi(x) = x + \frac{\gamma}{x}, \quad \gamma > 0, \quad (23)$$

where the condition put on the parameter  $\gamma$  leads to a SUSY potential  $\Phi$  which characterizes a shape-invariant SUSY pair of radial harmonic oscillator-like Hamiltonians with broken SUSY. Due to the condition (13) even the SUSY potential (12) with above  $\Phi$  gives rise to the radial harmonic oscillator potential

$$V_+(x) = \frac{x^2}{2} + \frac{\gamma(\gamma-1)}{2x^2} + \gamma + \frac{b+1}{2}, \quad (24)$$

which is exactly solvable and leads to the following spectral properties of  $H_+$ :

$$E_n^+ = 2n + 2\gamma + 1 + \frac{b}{2}, \quad \psi_n^+(x) = \left[ \frac{2n!}{\Gamma(n+\gamma+1/2)} \right]^{1/2} x^\gamma e^{-x^2/2} L_n^{(\gamma-1/2)}(x^2). \quad (25)$$

Here  $L_n^{(\nu)}$  denotes an associated Laguerre polynomial of degree  $n$  and index  $\nu$ . As in the previous case the general solution of (13) can be expressed in terms of confluent hypergeometric functions. Here, however, we are only interested in cases with broken SUSY and with this constraint the most general solution reads

$$u(x) = {}_1F_1\left(-\frac{b}{4}, \gamma + \frac{1}{2}, -x^2\right) = e^{-x^2} {}_1F_1\left(\gamma + \frac{b+2}{4}, \gamma + \frac{1}{2}, x^2\right). \quad (26)$$

This solution will be strictly positive if  $b > -4\gamma - 2$ . The corresponding SUSY partner potential reads

$$V_-(x) = \frac{x^2}{2} + \frac{\gamma(\gamma+1)}{2x^2} + \gamma - \frac{b+1}{2} + \frac{u'(x)}{u(x)} \left( 2x + \frac{2\gamma}{x} + \frac{u'(x)}{u(x)} \right) \quad (27)$$

and is plotted, for example, as Figure 5 in ref. [8]. The eigenvalues of the Schrödinger Hamiltonian  $H_-$  for this potential are identically to those of  $H_+$  given in (25). The

eigenfunctions can also be obtained from those in (25) via the SUSY transformation (4) and read

$$\psi_n^-(x) = \left[ \frac{2n!}{(n+\gamma+\frac{1}{2}+\frac{b}{4})\Gamma(n+\gamma+1/2)} \right]^{1/2} x^{\gamma+1} e^{-x^2/2} \left( L_n^{(\gamma+1/2)}(x^2) + \frac{u'(x)}{2xu(x)} \right). \quad (28)$$

Again, as in the previous case we can construct a drift potential with decay rates and modes given by the eigenvalues and eigenfunctions of  $H_-$ . This drift potential is explicitly given by

$$U(x) = \frac{1}{2}x^2 + \gamma \log x + \log u(x) = -\frac{1}{2}x^2 + \gamma \log x + \log [ {}_1F_1(\gamma + \frac{b+2}{4}, \gamma + \frac{1}{2}, x^2) ] \quad (29)$$

and because of broken SUSY does not have a stationary distribution. To be more explicit, for small  $x > 0$  it has a logarithmic “hole” at the origin, i.e.  $U(x) \approx -\gamma |\log x|$  for  $x \ll 1$ , whereas for large  $x \rightarrow \infty$  it becomes asymptotically that of a harmonic oscillator, i.e.  $U(x) \approx x^2/2$  for  $x \gg 1$ . In addition to that, for values of  $b$  close to its lower limit  $-4\gamma - 2$  this drift potential exhibits a local minimum. In other words, it is metastable. In fact, the smallest decay rate is given by  $E_0^- = 2\gamma + 1 + b/2 \searrow 0$  for  $b \searrow -4\gamma - 2$ . For larger values of  $b$  this local minimum disappears and the drift potential (29) becomes unstable. A typical plot of this potential is given in Figure 1 where we have shown (29) for  $\gamma = 1$  and  $b \in (-6, -5)$ . Note that in Figure 1 we have plotted  $U(x)$  versus  $\exp\{-x\}$ .

## 5 Final remarks

In this paper we have extended a recent approach to the construction of drift potentials for which the associated Fokker-Planck equation can be solved exactly. As starting point we have chosen a SUSY potential  $\Phi$  which generates a pair of shape-invariant quantum mechanical potentials leading to exactly solvable quantum Hamiltonians. This SUSY potential can be perturbed (by adding  $u'/u$ ) in such a way that one of these Hamiltonians,  $H_+$ , remains in the class of shape-invariant exactly solvable Hamilton operators. The partner Hamiltonian  $H_-$ , however, is a new one and due to SUSY its spectral properties can be obtained from  $H_+$  in a straightforward way. Using the close relations between SUSY quantum mechanics and the Fokker-Planck equation we have used these results to find also new drift potentials with exactly known decay rates and modes. In this paper we have constructed stable as well as unstable drift potentials associated with unbroken and broken SUSY, respectively. The examples for unbroken SUSY are actually not new and have already been studied by Hongler and Zheng [9]. However, the present results for broken SUSY leading, in particular, to metastable drift potentials are new. To our knowledge, these are the first (on the positive half line) analytical metastable drift potentials for which the decay rates and modes can be given in closed analytical form. Besides some practical applications these metastable potentials can also serve as a testing ground for approximation methods. Recall that for metastable drift potentials the

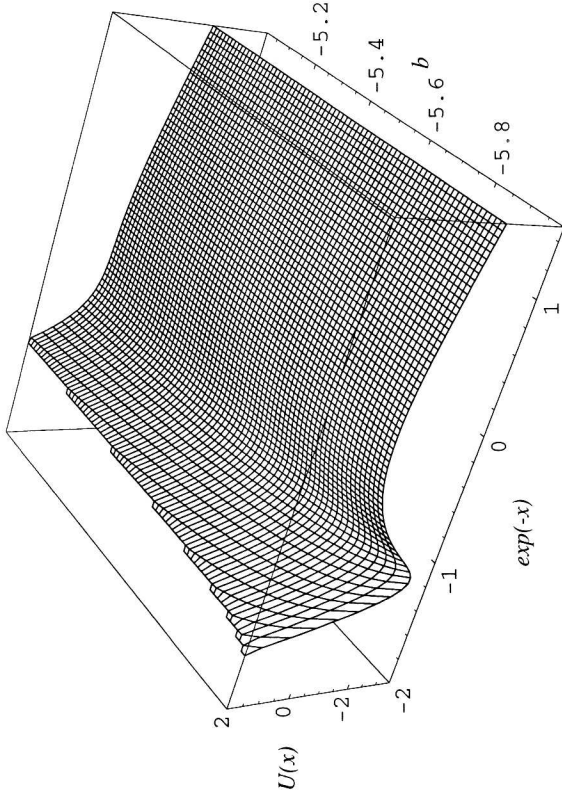


Figure 1: A family of drift potentials (29) for  $\gamma = 1$  showing a transition from metastability to instability with increasing  $b$ . The corresponding decay rates and decay modes are known in closed form. Note that we have plotted  $U(x)$  versus  $\exp\{-x\}$ .

fluctuation operator for the classical “bounce” solution has a negative eigenvalue and, therefore, leads to serious singularities within a saddle-point approximation [13]. The standard treatment in such unstable situations goes back to Langer [14] and is based on some analytical continuation techniques. These formal treatments can now be applied to and (for the first time) tested with the examples presented in Section 4.2.

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